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Frege or Dedekind? Towards a Reevaluation of Their Legacies

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The philosophy of mathematics has long been an important part of philosophy in the analytic tradition, ever since the pioneering works of Frege and Russell. Richard Dedekind was roughly Frege's contemporary, and his contributions to the foundations of mathematics are widely acknowledged as well. The philosophical aspects of those contributions have been received more critically, however. In the present chapter, Dedekind's philosophical reception is reconsidered. At the chapter's core lies a comparison of Frege's and Dedekind's legacies, within and outside of analytic philosophy. While the comparison proceeds historically, it is shaped by current philosophical concerns, especially by debates about neo-logicist and neo-structuralist views. In fact, philosophical and historical considerations are intertwined thoroughly, to the benefit of both. The underlying motivation is to rehabilitate Dedekind as a major philosopher of mathematics, in relation, but not necessarily in opposition, to Frege.

The chapter is structured as follows: In Section 1, a brief reminder about Frege's and Dedekind's contributions will be provided, together with a look at how they saw the relationship between their works themselves. In Section 2, we will turn to the early reception each received in analytic philosophy, from Russell on, with the focus on critical responses to Dedekind. Then, in Section 3, the revival of Frege's ideas since the 1950s, the rise of neo-logicism since the 1980s, and further criticisms of Dedekind within those contexts will be discussed. In Section 4, after noting the more positive response Dedekind received in mathematics, I will bring to bear the rise of neo-structuralism since the 1980s, thereby starting to turn the tables. This will be followed, in Section 5, by more direct defenses of Dedekind, to be found in Ernst Cassirer's discussions of his works and in current philosophy of mathematics. The chapter will end with some reflections on where this leaves us, with respect to Frege, Dedekind, and their philosophical legacies.

1 Frege, Dedekind, and their Relationship

Most of Dedekind's philosophical remarks can be found in two small booklets, *Stetigkeit und irrationale Zahlen* (1872) and *Was sind und was sollen die Zahlen?* (1888). They were published during the same period as Frege's main works, *Begriffsschrift* (1879), *Die Grundlagen der Arithmetik* (1884), and *Grundgesetze der Arithmetik* (1893/1903).[1] There is quite a bit of overlap between these texts. Both authors present new foundations for the theories of the natural and real numbers; they both proceed without relying on geometry or, more generally, any 'intuitive' assumptions; and they present 'logicist' alternatives instead, based on new theories of relations, functions, and classes. There are further similarities with respect to details. For instance, the ways in which they analyze mathematical induction logically – Frege in terms of the 'ancestral' relation, Dedekind in terms of the notion of 'chain' – are not only equally innovative but equivalent.[2]

Besides such similarities there are also differences. A commonly mentioned one is that Dedekind's foundational contributions lie more on the model-theoretic side (studying, e.g., models of theories and isomorphism results), while Frege's are primarily proof-theoretic (based on his new proof system). A more general difference is that, while Frege produced some mainstream mathematical works besides his trailblazing contributions to mathematical logic, they remained minor. Dedekind, in contrast, was a major, highly influential contributor to mathematics, especially to algebra and number theory.[3] With respect to philosophy, the situation is reversed. Frege wrote extensively on philosophical topics, in ways that had a strong impact over time; but only a few philosophical remarks are sprinkled though Dedekind's writings. Still, an important difference between them, for present purposes, concerns a philosophical matter. Namely, Dedekind articulated a structuralist view about the nature of mathematical objects, based on certain kinds of 'abstraction' and 'free creation'; Frege constructed his logical objects in a non-structuralist way, as explicitly defined equivalence classes.[4]

I will explore both the similarities and the differences further as we go along. But let me address another question first: How did Frege and Dedekind perceive their relationship themselves? The two thinkers never met in person; nor did they have a correspondence, as far as I know. It is also evident that they developed their basic ideas independently. Thus, in the Preface to the second edition of *Was sind und was sollen die Zahlen?* (published in 1893) Dedekind remarks that it was only 'about a year after the publication of my memoir [that] I became acquainted with G. Frege's *Grundlagen der Arithmetik*' (Dedekind, 1963, p. 42). Dedekind does not say anything about Frege's *Begriffsschrift* here; but since he had settled on his core ideas already before its publication, he clearly developed them independently.[5] Frege mentions Dedekind's works that predate his own, such as *Stetigkeit und irrationale Zahlen*, neither in *Begriffsschrift* nor in *Grundlagen*; and in the later *Grundgesetze* his disagreements with Dedekind predominate.[6]

After having become aware of each other's writings, both Frege and Dedekind commented on the relation between their projects. Above, I quoted from Dedekind's only explicit reference to Frege in print, in the second edition of *Was sind und was sollen die Zahlen?* (1893). He continues:

> However different the view of the essence of number adopted in [Frege's *Grundlagen*] is from my own, it contains, particularly from section 79 on, points of very close contact with my paper, especially with my definition (44) [of the notion of chain]. The agreement, to be sure, is not easy to discover on account of the different form of expression; but the positiveness with which the author speaks of the logical inference from n to $n+1$ [...] shows plainly that here he stands upon the same ground with me. (Dedekind 1963, pp. 42–43)

Dedekind does hint at some differences to Frege in this passage (more on those below). But his emphasis on positive connections between their approaches is typical for him. (His response to, say, Cantor's rival theory of real numbers is similar.)

Equally typical for Frege is that his published reactions to Dedekind's works, in both volumes of *Grundgesetze*, are strongly critical. Yet they are not entirely negative. In the Preface to Volume I of *Grundgesetze*, Frege calls Dedekind's essay on the natural numbers 'the most thorough work on the foundations of arithmetic that has come to my attention in the last few years' (Frege, 1893, p. 196). He also sees an agreement with respect to their basic convictions, since 'Dedekind too is of the opinion that the theory of numbers is a part of logic' (ibid.). Indeed, in the original Preface of *Was sind und was sollen die Zahlen?* Dedekind had talked about developing 'that part of logic which deals with the theory of numbers' as his goal, then adding:

> In speaking of arithmetic (algebra, analysis) as a part of logic I mean to imply that I consider the number-concept entirely independent of the notions of intuition of space and time, that I consider it an immediate result from the laws of thought. (Dedekind, 1963, p. 31)

It is based on such programmatic statements, together with corresponding technical details, that Frege could acknowledge Dedekind to be a fellow 'logicist'.

While Frege does see connections between his and Dedekind's projects, he couples their acknowledgement with a battery of criticisms. His main criticisms in Volume I of *Grundgesetze* concern *Was sind und was sollen die Zahlen?* Frege's first such criticism (still in the Preface) is the following:

While Dedekind is also pursuing a logicist project, the conciseness of his proofs – the fact that they are 'merely indicated', not 'carried out in full' (Frege, 1893, p. 196) – does not allow one to be sure that all presuppositions have been identified. This problem is aggravated by the fact, also pointed out by Frege, that Dedekind does not formulate his basic laws explicitly; much less does he provide a complete list of them. Consequently, it is not clear why Dedekind's theory of 'systems' should be seen as logical. Altogether, it is thus questionable whether a logicist reduction of arithmetic has actually been achieved.

Further criticisms raised by Frege in Volume I of *Grundgesetze* (now its Introduction) concern details of what Dedekind does say about 'systems'. Let me mention two of them. First, Frege sees Dedekind's treatment of systems with one element (his identification of singletons with their elements) as problematic because, among others, it encourages confusing the element and subset relations. Together with his exclusion of the empty system, it also makes one wonder whether Dedekind thinks of systems as 'consisting' of their elements (like mereological sums), a view Frege rejects strongly. Second, while Dedekind conceives of systems extensionally – as Frege notes approvingly – some of his remarks about them are problematic. In particular, when Dedekind writes about 'regarding [various objects] from a common point of view', this makes it appear as if what underlies the existence of systems is some mental operation of 'putting together in the mind'. Their nature and existence thus become too subjective, while Frege insists on their objectivity. In short, Dedekind's position on systems seems 'psychologistic'.

In later parts of this chapter, I will defend Dedekind against a number of criticisms, including the charge of psychologism. But let me formulate initial evaluations of Frege's other charges, as just mentioned, right away. Arguably, several of them are based on an uncharitable reading. Thus, Dedekind explicitly acknowledges the possibility of introducing an empty system; he works with a clear distinction between elements and subsets elsewhere; and he treats systems in an abstract (non-mereological) way in general. Yet, with his first main charge Frege did put his finger on a sore spot. While one can defend Dedekind's way of 'sketching' proofs as acceptable from a usual mathematical point of view, he does in fact smuggle in unnoticed presuppositions and use unstated laws at times. For example, in his treatment of infinity, the Axiom of Choice is used implicitly (as Zermelo pointed out later). More basically, it is hard to be sure what exactly Dedekind's conception of 'systems' is; even more so for 'logic'.[7]

While Frege formulates these criticisms of *Was sind und was sollen die Zahlen?* in *Grundgesetze*, Volume I, it is also noteworthy which ones he does not raise. Let me again mention two: First, he does not object to Dedekind's notions of 'abstraction' and 'free creation' here (although he will do so later). Second, Frege does not bring up the part of Dedekind's essay that would soon became most infamous: his 'proof' of the existence of an infinite system (Proposition 66), and more specifically, the appeal to 'the totality of all things which can be objects of my thinking' in it (Dedekind, 1963, p. 64). While not mentioned in *Grundgesetze* at all, there is another place where Frege addresses that appeal, however: his posthumously published 'Logic' (drafted in 1897). In that piece, Frege defends his usual distinction between objective 'thoughts' and subjective 'acts of thinking'. After acknowledging that this involves a non-standard use of the word 'though', he points out that there are others who use it similarly – including Dedekind. More specifically, he argues: As we may assume that Dedekind 'has not thought infinitely many thoughts', he, too, must use 'thinking' in a non-psychologistic way. Note two aspects here: Not only does Frege not criticize Dedekind for holding psychologistic views in this context; he also rejects neither Proposition 66 nor its proof.

As we saw, Frege's main criticisms of Dedekind in Volume I of *Grundgesetze* are directed against the theory of 'systems' from *Was sind und was sollen die Zahlen?*, which he had clearly studied carefully by that time. Those in Volume II of *Grundgesetze* concern the earlier essay, *Stetigkeit und irrationale Zahlen*, and more particularly, Dedekind's introduction and famous characterization of the real numbers in it. Frege starts again with a positive remark in this context. He commends Dedekind for implicitly rejecting formalist view – by making an explicit distinction between signs and what they stand for, by treating the real numbers as objects referred to by means of signs, and by conceiving of equality for numbers in a corresponding 'objectual' way, all details Frege agrees with. But then his critical assault resumes.

Frege's first criticism at this point concerns the following: He observes that Dedekind, after introducing his system of cuts on the rational numbers, does not identify the real numbers with the cuts; rather, he talks about the 'creation' of new objects, one for each cut. Frege's objection is not that the notion of 'creation' at play here is psychologistic. Nor is it that there cannot be such objects, with only structural properties. Rather, he points out that Dedekind has not inquired generally into when such 'creation' is feasible, including whether there are any limits to it. One obvious limit is when one is led to an inconsistency, a case he accuses Dedekind of ignoring. Frege then groups him with other thinkers, such as Hermann Hankel and Otto Stolz, who use 'creative definitions' without any justification, concluding sarcastically: '[T]he inestimable advantage of a creative definition is that it saves us a proof'. But this charge against Dedekind can again be deflected, since it ignores the role Dedekind's construction of the system of cuts plays for him (similarly for Proposition 66 in Dedekind's treatment of the natural numbers).[8]

Frege's second main criticism of Dedekind in Volume II of *Grundgesetze* is the most subtle but also the most slippery. Just before admitting, rather surprisingly, that his own introduction of extensions via Basic Law V might

perhaps be seen as a kind of 'creation' as well (although expressly not as a 'definition'), Frege declares:

> If there are logical objects at all – and the objects of arithmetic are such objects – then there must be a means of apprehending, or recognizing, them. This service is performed for us by the fundamental law of logic that permits the transformation of an equality holding generally into an equation [i.e., Basic Law V]. Without such a means a scientific foundation for arithmetic would be impossible. (Frege, 1903, pp. 278–279)

The criticism of Dedekind's procedure is, thus, that he does not provide us with a 'means of apprehending or recognizing' for the novel objects he introduces. One intriguing aspect here is the connection to the well-known 'Julius Caesar problem', as brought up in Frege's *Grundlagen*. Another is that Dedekind, if read charitably, does actually provide the required 'means', albeit implicitly. Namely, his structurally conceived numbers have only 'arithmetic' properties, which differentiates them from objects like Julius Caesar. Perhaps this Fregean charge can therefore be deflected as well. [9]

2 Russell's Criticisms of Dedekind and their Immediate Impact

In *Grundgesetze*, Frege expressed frustration about the lack of attention his works had received so far. Dedekind's two foundational essays were also not widely appreciated initially, especially by philosophers.[10] One of the first to pay careful attention to both was Bertrand Russell. Most famous in this connection is, of course, Russell's discovery of the antinomy named after him, which applies to Frege's and Dedekind's theory of classes. The fact that the Russell class (of all classes that do not contain themselves) can be formed according to these theories, thus leading immediately to a contradiction, confirmed Frege's concerns about consistency in the worst possible way. After being told about it by Russell in 1902, his response – in an Appendix to Volume II of *Grundgesetze* – showed consternation. Obviously, there was a problem with his Basic Law V. But without it, how could arithmetic be 'scientifically established'? When Dedekind found out, already in 1899, about antinomies like Russell's from Georg Cantor (who had discovered them independently), he was equally dismayed. According to one report, he wasn't sure any more whether 'human thinking was really rational'.[11]

Besides the devastating impact of his antinomy, Russell's more general reception of both Frege's and Dedekind's writings is crucial, especially for us, in two other respects as well. First, it was with Russell's writings that a now entrenched view of 'logicism' emerged, one that gives pride of place to Frege and Russell while tending to exclude Dedekind. Second, it was through Russell's works, together with those of his students and successors,

that Fregean ideas became a central part of the analytic tradition, while Russell's criticisms of Dedekind led to his relative neglect by philosophers. In the rest of this section and the next, I will elaborate on both of these points. I will also provide a brief summary of Russell's further criticisms of Dedekind.

I already noted that, despite his own criticisms, Frege saw Dedekind as a fellow logicist. Actually, he was widely recognized as such in the late nineteenth century – a number of writers, from C.S. Peirce through Ernst Schröder to the early Hilbert, saw in Dedekind a main, and perhaps the original, 'logicist'.[12] This changed in the twentieth century. Why? Several factors played a role, perhaps most importantly the following: After the discovery of his and related antinomies, Russell's response to them, as worked out in *Principia Mathematica* (1910–1913), became the primary logicist option. Indeed, it came to be seen as its paradigm case, thus as almost definitional of 'logicism'. Moreover, *Principia* was clearly more a successor to Frege's theory than to Dedekind's (with its explicit logical laws and its deductive emphasis). This is also how Russell viewed the matter, including in some retrospective accounts. Dedekind's approach, in contrast, came to be seen as a predecessor to axiomatic set theory, to model theory, and to Hilbertian formalism (in striking contrast to Frege's praise of Dedekind as an anti-formalist).[13]

What were Russell's criticisms of Dedekind, besides his antinomy? Like in Frege's case, let me go over several main ones. In *Principles of Mathematics* (1903), his first relevant book, Russell too starts out positively, by acknowledging several 'brilliant contributions' by Dedekind (as well as by Cantor, Frege, and Peano). These include: Dedekind's general treatment of relations, including the notion of 'progression' (Dedekind's 'simple infinity'); his corresponding notion of 'chain' and analysis of mathematical induction (which Russell took over mainly from Dedekind, not from Frege, as Quine pointed out later); his definition of infinity; and his use of cuts for introducing the real numbers. Again like Frege, Russell then added various negative points, in *Principles* and later texts. These concern Dedekind's treatment of both the natural and the real numbers.

It appears that Russell struggled from the beginning with getting a good, or even any, handle on Dedekind's structuralist position. He remarks that Dedekind prefers to view the natural numbers as 'ordinals', not as 'cardinals'. One initial, vague complaint is, then, that ordinals are more 'complicated' than cardinals. Russell continues:

> Now it is impossible that this account should be quite correct. For it implies that the terms of all progressions other than the ordinals are complex, and that the ordinals are elements in all such terms, obtained by abstraction. But this is plainly not the case. A progression can be formed of points or instants, or of transfinite ordinals, or of cardinals,

> in which, as we shall shortly see, the ordinals are not elements. (Russell, 1903, pp. 248–249)

What Russell seems to claim in this passage is that the entities ('terms') in any simple infinity ('progression') must contain Dedekind's ordinal numbers 'as elements'; and he rejects the latter as false. But how is that related to Dedekind's position? The fact that Russell struggles in this regard comes through further when he writes: 'What Dedekind intended to indicate was probably a definition by means of the principle of abstraction, such as we attempted to give in the preceding chapter' (p. 249). It seems that the only way for Russell to make sense of Dedekind's 'abstraction' was to assimilate it to his own 'principle of abstraction'. Yet, Dedekindian abstraction works quite differently.[14]

Russell's second main objection, which follows immediately after the first, concerns Dedekind's corresponding structuralist conception of mathematical objects:

> Moreover it is impossible that the ordinals should be, as Dedekind suggests, nothing but the terms of such relations as constitute a progression. If they are to be anything at all, they must be intrinsically something; they must differ from other entities as points from instants, or colours from sounds. (ibid.)

Here the charge is that there cannot be entities as conceived of by Dedekind. According to Russell, every 'term', 'entity', or 'object' simply has to have non-structural properties. This seems to be a fundamental ontological conviction, or prejudice, for him – it is not justified further. Next, Russell is led to the following suggestion:

> What Dedekind presents to us is not the numbers, but any progression alike, and his demonstration nowhere – not even where he comes to cardinals – involve any property distinguishing numbers from other progressions. (ibid.)

What Russell suggests in this passage is that, along Dedekind's lines, any statement about numbers is really a statement about all 'progressions', i.e., it should be understood in terms of a universally quantified proposition. Russell's attribution of this 'universalist' position to Dedekind – seemingly in an attempt to be charitable – again misses its mark. However, it turned out to be quite influential later on (as we will see below).[15]

Let us move on to Russell's criticisms of Dedekind concerning the real numbers, in *Principles* and later. In this context, too, Russell makes some claims that are puzzling. For example, it is difficult to see how one can find a clearer analysis of the notion of 'continuity' in Cantor's writings

compared to Dedekind's; but that is what Russell maintains. He also raises the following objection: For Dedekind, the existence of the real numbers remains a 'sheer assumption', i.e., it is not backed up by argument. Like Frege, Russell lumps Dedekind together with other writers in this connection, namely ones that simply 'postulate' the existence of mathematical entities. And again like Frege, he has only scorn and ridicule for such views. As he famously puts it later:

> The method of 'postulating' what we want has many advantages; they are the same as the advantages of theft over honest toil. Let us leave them to others and proceed with our honest toil. (Russell, 1919, p. 71)

I already gave a response to this kind of charge above. Namely, it ignores Dedekind's construction of the system of cuts before introducing the real numbers; similarly for his explicit attempt to establish the existence of a simple infinity (Proposition 66). In other words, Dedekind does provide some 'honest toil' in this connection.

But perhaps Russell's most interesting comment on Dedekind concerns exactly the 'proof' of Proposition 66. It helps to be a bit more explicit about it now. Dedekind does not just appeal to 'the totality S of all things which can be objects of my thinking' in it; he also brings in his own 'ego', or 'self', as a distinguished element, and the function that maps a thought s to 'the thought s', that s can be object of my thought' (Dedekind, 1963, p. 64). The argument is, then, that the collection of all the successors of the distinguished element under that function (the corresponding 'chain') forms an infinite system. Now, in *Principles* Russell first notes the similarity of this argument to one provided in Bernard Bolzano's *Paradoxien des Unendlichen* (as does Dedekind in the second edition of his essay). He then reconstructs the Bolzano-Dedekind argument as follows:

> For every term or concept there is an idea, different from that of which it is the idea, but again a term or concept. On the other hand, not every term or concept is an idea. There are tables, and ideas of tables; numbers and ideas of numbers; and so on. Thus there is a one-one relation between terms and ideas, but ideas are only some among terms. Hence there is an infinite number of terms and of ideas. (Russell, 1903, p. 307)

What is the problem, then? Is Russell's criticism that, if understood in a mental or psychological sense, there may not exist enough 'ideas' for Dedekind's purposes? Not exactly, since he adds the following in a footnote: 'It is not necessary to suppose that the ideas of all terms exist, or form part of some mind; it is enough that they are entities' (ibid.). So far, this is not a strong objection, if any, to Dedekind's proof.[16]

In Russell's article, 'The Axiom of Infinity' (published in 1904, a year after *Principles*), a new twist is added to this line of thought. Russell asks us to consider the following sequence: 0 = the number of the empty class; 1 = the number of {0}; 2 = the number of {0, 1}, etc. He notes that the entities introduced along such lines – the finite cardinal numbers – are all different; and there are entities different from all of them, such as 'the number of all finite cardinal numbers', i.e., the first infinite cardinal number. What we get, then, is a proof of the existence of an infinite class that is parallel to Dedekind's and Bolzano's but avoids using the notion of 'idea'. Why might such a modified proof be preferable for Russell? Because it provides 'a strict proof appropriate to pure mathematics, since the entities with which it deals are exclusively those belonging to the domain of pure mathematics' (pp. 257–258). This leads to the following criticism of Dedekind:

> Other proofs, such as the one from the fact that the idea of a thing is different from the thing, are not appropriate to pure mathematics, since they [...] assume premises not mathematically demonstrable. (Russell, 1904, p. 258)

In other words, for Russell (the Russell of this early period) the problem is not that we cannot get a proof of the existence of an infinite class by appealing to 'ideas'. In fact, he adds: 'Such proofs are not on that account circular or otherwise fallacious' (ibid.). It is, rather, that they involve a dimension 'not appropriate to pure mathematic'.

Russell's variant of the Bolzano–Dedekind proof works only if we have the operator 'the number of...' at our disposal. When writing his 1904 article, Russell seems to still think that his initial conception of the natural numbers, as equivalence classes of classes (essentially Frege's from *Grundgesetze*), provides what is needed here. Moreover, in *Principles of Mathematics* the following related remark occurs: 'There seems, in fact, to be nothing to choose, as regards logical priority, between ordinals and cardinals, except that the existence of the ordinals is inferred from the series of the cardinals' (Russell, 1903, p. 241). But with the collapse of Russell's early theory of classes this option vanishes. His response is to replace that theory by a 'no-classes theory of classes', within a ramified theory of types. The existence of an infinite class (at one type level) is then no longer provable. At that point, Russell adopts an axiom of infinity (for individuals), most prominently in *Principia Mathematica*, since no other option seems available. But what is the status of that axiom? In particular, can it be seen as a logical axiom?

By the time of *Introduction to Mathematical Philosophy* (1919), Russell has come to acknowledge that he has a basic problem in this connection: his axiom of infinity, while not contradictory, is 'not demonstrably logical' (ibid., p. 141). This leads him back to (his version of) Dedekind's original 'proof', which he now criticizes as follows:

> If the argument is to be upheld, the 'ideas' intended must be Platonic ideas laid up in heaven, for certainly they are not on earth. But then it at once becomes doubtful whether there are such ideas. If we are to know that there are, it must be on the basis of some logical theory, proving that it is necessary to a thing that there should be an idea of it. We certainly cannot obtain this result empirically, or apply it, as Dedekind does, to 'meine Gedankenwelt' – the world of my thoughts. (Russell, 1919, p. 139)

As the subsequent discussion makes clear, Russell now doubts whether we can assume the existence of an 'idea' corresponding to every object. In fact, he has become skeptical about the very notion of 'idea'. As he puts it: 'It is, of course, exceedingly difficult to decide what is meant by "idea"' (ibid.). The basic problem with Dedekind's procedure remains, however, that no 'logical theory' can assure us of what is needed in it.

Clearly, Russell was quite critical of Dedekind's philosophical views, as opposed to his technical achievements. On the other hand, he had high praise for Frege as a philosopher, from *Principles* on. Both reactions proved hugely influential. Let me illustrate that fact by considering three of Russell's main heirs briefly: Ludwig Wittgenstein, Rudolf Carnap, and W.V.O. Quine. In Wittgenstein's *Tractatus Logico-Philosophicus* (1921/1922), the non-logical nature of Russell's axiom of infinity is pointed out; Russellian logicism is thus rejected. Nevertheless, the *Tractatus* is deeply influenced, not only by Russell, but also by Frege. In contrast, Dedekind is not mentioned at all in the text. In Wittgenstein's later writings, Russell- and Frege-inspired topics remain central. Dedekind now comes up occasionally as well, for example, in the *Remarks on the Foundations of Mathematics* (1956), but in highly critical, even dismissive terms. Among others, Wittgenstein criticizes Dedekind's theory of the real numbers along finitist and constructivist lines.

Carnap was another of Russell's main heirs. He was also strongly influenced by the *Tractatus*, at least for a while. Carnap does not challenge the significance of Dedekind's technical achievements, as Wittgenstein seems to do. But like Wittgenstein, he engages much more with Frege's philosophical views than with Dedekind's, as works such as *Meaning and Necessity* (1949) illustrate. In Carnap's very influential article on logicism, 'The Logical Foundations of Mathematics' (1931), he also further entrenches the view that Frege and Russell were the two main founders of logicism, while Dedekind hardly matters. Similar remarks apply to Quine, Russell's third main heir. In Quine's works on logic and the foundations of mathematics, there are numerous references to Dedekind's mathematical results, which

are taken for granted. Yet, Frege is mentioned much more frequently; and Dedekind is usually not engaged as a philosopher.

3 Frege Revivals, Neo-Logicism, and Further Criticisms of Dedekind

Frege was valued highly, as a philosopher, by several of the most influential figures in the analytic tradition, as we just saw. Nevertheless, his writings were not read widely until the 1950s, especially in the English-speaking world. This changed with the publication of several new translations of his works, including J.L. Austin's English rendering of *Grundlagen der Arithmetik* (1950), and Peter Geach and Max Black's collection, *Translations from the Philosophical Writings of Gottlob Frege* (1952). Characteristically, work on the latter was strongly supported by both Russell and Wittgenstein. The parallel impact of Carnap and Quine in the U.S. is reflected, among others, in Paul Benacerraf and Hilary Putnam's influential collection, *Philosophy of Mathematics: Selected Readings* (first published in 1964). It contains substantive excerpts from texts by Frege and Russell, Carnap's article on logicism mentioned above, and several pieces by Quine – but nothing by Dedekind.[17]

From the 1960s on, the philosopher who contributed most to the revival of Fregean ideas was Michael Dummett. His highly influential book, *Frege: Philosophy of Language*, was published in 1973, after having circulated in manuscript form earlier. Its author had set himself the task of providing not only an exegesis of Frege's views on logic and language, but also a thorough, more general exploration of Fregean topics. Dummett's book appeared during a period when the philosophy of language was quickly becoming the central sub-discipline of analytic philosophy (partly due to Wittgenstein's, Carnap's, and Quine's influence). Consequently, Dummett's discussion of Frege led to widespread debates about his corresponding views, especially the sense-reference distinction. And even reactions against Frege in that connection, as provided by, for example, Saul Kripke and John Perry, consolidated his status as one of the 'founders' of the analytic tradition.

From early on, Dummett had meant to supplement his first book by another on Frege's philosophy of mathematics; but its publication was long delayed. In *Frege: Philosophy of Language*, some relevant topics were covered, including questions about abstract objects and identity. Dummett even claimed that it was Frege's work 'which inaugurated the modern period in the philosophy of mathematics' (ibid., p. 656). But it was the writings of one of Dummett's students, Crispin Wright, which led to a revival of Fregean views about mathematics in the 1980s. Crucial here was the publication of Wright's book, *Frege's Conception of Numbers as Objects* (1983). Its Preface starts as follows:

> In the middle and later years of this century Frege's ideas on a wide class of issues in the philosophy of language have assumed a deserved centrality in the thinking of philosophers interested in that area. Of his philosophy of mathematics, in contrast, it is fair to say that its felt importance to contemporary work remains largely historical (p. ix).

Like Dummett, Wright was not really interested in historical aspects in his book. Instead, he wanted to provide a rational reconstruction of Frege's approach to mathematics, one that established its continuing relevance (parallel to Frege's by then classical approach to language). In Wright's own words, the goal was 'to revitalize discussion of the questions [in the philosophy of mathematics] to which Frege's constructive effort was aimed, and of his specific answers' (ibid., p. x). Crucial for this purpose was to find a way around the problem that seemed to still doom a Fregean approach: its inconsistency.

Building on Dummett's remarks about abstract objects, identity conditions, and the use of singular terms, Wright went further than him in defending Fregean 'platonism' about mathematical objects. He soon found an ally in Bob Hale, whose book, *Abstract Objects* (1987), added to the defense on the epistemological side. Together they launched an influential 'neo-logicist' research program. As that program is well known today, I will not recapitulate its details here.[18] But let me provide reminders about a few core ideas that will be relevant for us. The central technical result – 'Frege's Theorem' – establishes that all of arithmetic can be derived (in second order logic) from 'Hume's Principle':

$\#Fs = \#Gs \Leftrightarrow F$ and G can be mapped 1–1 onto each other[19]

Frege had formulated this principle but had not treated it as a basic law. Instead, he tried to derive it from his theory of classes (and corresponding definitions). Wright's new suggestion is to drop that problematic theory and start with Hume's Principle itself.

This 'neo-Fregean' suggestion is attractive because the resulting theory – 'Frege Arithmetic' – can be shown to be (relatively) consistent, i.e., not subject to Russell's antinomy. It can also be generalized by adding other 'abstraction principles', e.g., to ground the theory of real numbers. Beyond that, Wright and Hale argued that what results should count a form of logicism. Their argument in the simplest case, that of the natural numbers, is this: Frege Arithmetic relies solely on a principle of numerical identity, encapsulated in Hume's Principle, that is 'quasi-definitional', or in some sense 'constitutive', of the concept of cardinal number. The latter view remains controversial, however. The most interesting, but again controversial, aspect for present purposes is that such a neo-logicist approach appears to allow for a proof of the existence of many abstract objects, such as the infinite sequence of natural numbers.[20]

Various aspects of the neo-logicist program have been called into question by now; thus, its philosophical significance remains in doubt. Nevertheless,

Wight, Hale, and their co-workers clearly succeeded in reviving Fregean questions and answers, or broadly Fregean approaches, in the philosophy of mathematics, which now form an established part of the philosophy of mathematics in the analytic tradition. Besides their aim to rehabilitate Frege, what unites many neo-Fregeans is a critical attitude toward Dedekind. Georg Boolos makes that attitude explicit when he declares:

> One of the strangest pieces of argumentation in the history of logic is found in Richard Dedekind's *Was sind und was sollen die Zahlen?*, where, in the proof of that monograph's Theorem 66, Dedekind attempts to demonstrate the existence of an infinite system. (Boolos, 1998, p. 202)

Why is Dedekind's argument so exceedingly 'strange'? The reason is that it starts with 'as wildly non-mathematical an idea as his own ego' (ibid.). Boolos' remark clearly echoes one of Russell's criticisms. But it is striking how much less charitable, and more rhetorically charged, his formulation is than Russell's, even if their final conclusion is similar.

Within the neo-Fregean literature, the most detailed criticism of Dedekind can be found in Michael Dummett's later book, *Frege: Philosophy of Mathematics*, which finally appeared in 1991. It contains a whole chapter in which Frege's and Dedekind's approaches are compared explicitly; and further relevant remarks are sprinkled throughout the text. Dummett is as polemical as Boolos, as we will see. Also like Boolos, he repeats points raised by Russell against Dedekind; but he also adds new criticisms, presented in a 'Fregean' spirit. Dummett starts by acknowledging that Dedekind provided valuable contributions to issues Frege barely touched on, with his recursive treatments of addition, multiplication, and exponentiation for the natural numbers. His real sympathies start coming through, however, when he states:

> There is indeed a significant contrast between the contemporary but independent work of Frege and Dedekind on the foundations of number theory; the difference could certainly be characterized by saying that Dedekind's approach was more mathematical in nature, Frege's more philosophical. (Dummett, 1991, p. 11)

Compared to Dedekind's works, Dummett characterizes Frege's *Grundlagen der Arithmetik* – which he praises as his masterpiece – as 'by far the more philosophically pregnant and perspicacious'. Once again, Frege is valued much higher. But to his credit, Dummett does engage Dedekind as a philosopher in what follows.

What are Dummett's main criticisms of Dedekind? The first one is familiar by now, from both Russell and Boolos, namely: Dedekind's alleged proof that infinite systems exist is based on 'a piece of non-mathematical reasoning' (p. 48). Dummett's second major criticism concerns Dedekind's view that 'abstraction' and 'free creation' are crucial for explaining what the natural and real numbers are. However, the objection here is not, along Russellian lines, that this involves a case of 'theft'. As Dummett admits:

> The case [...] is quite different from one in which a mathematician postulates a system of numbers satisfying certain general conditions. Dedekind provided a totality, composed of classes of rationals with which the real numbers could be correlated one to one; he had done all the honest toil required (p. 250).

Instead, Dummett argues that Dedekind's procedure 'leads to solipsism' (ibid.); or at the very least it tempts us, misleadingly, 'to scrutinize the internal operations of our minds' (p. 311). Connected with the latter point, Dedekind is again placed in 'bad company':

> It was virtually an orthodoxy, subscribed to by many philosophers and mathematicians, including Husserl and Cantor, that the mind could, by this means, create an object or system of objects lacking the features abstracted from, but not possessing any others in their place (ibid., p. 50).

To be more precise, Dummett acknowledges that Dedekind's position is different from Husserl's and Cantor's insofar as he speaks not of 'creating' individual numbers but whole systems of numbers by 'abstraction'. Yet, that difference is brushed aside when he concludes: 'Frege devoted a lengthy section of *Grundlagen*, sections 29–44, to a detailed and conclusive critique of this misbegotten theory' (ibid.).

As Dummett appeals to Frege as his authority in this context, it is worth pausing for a moment. As pointed out above, Frege does actually *not* voice this objection to Dedekind. It is true that he criticizes his psychologistic-sounding language concerning the notion of 'system'. But with respect to 'abstraction' and 'creation', Frege argues instead that Dedekind does not investigate the conditions and limits of his procedure enough, including formulating basic principles for it. This Fregean objection deserves a careful response (more on it below), while the one expressed by Dummett looks more like a 'criticisms by association', coupled with dismissive rhetoric. (As we saw, both Frege and Russell also used such strategies at points.) Moreover, Dummett seems far less charitable to Dedekind than Frege, just like Boolos was less charitable than Russell.

A third Dummettian objection, directed at the results of Dedekind's use of 'abstraction' and 'free creation', leads us back to Russell as well. It concerns the view that these operations result in objects with only relational or structural properties. After lauding *Principles of Mathematics* as Russell's 'great book of 1903', Dummett points to Russell's claim that 'it is impossible that the ordinals should be, as Dedekind suggests, nothing but the terms of such relations as constitute a progression'. He comments sympathetically: 'Russell

is here obstinately refusing to recognize the role assigned by Dedekind to the process of abstraction' (ibid., p. 50). Then he adds:

> [Dedekind] believed that the magical operation of abstraction can provide us with specific objects having only structural properties: Russell did not understand that belief because, very rightly he had no faith in abstraction thus understood (p. 52).

Why exactly was Russell right in opposing Dedekindian 'abstraction'; or why can't there be such objects? Russell's opposition seemed to be based simply on a ontological prejudice, as noted above. All Dummett has added, so far, is more rhetoric. But to be fair, he then provides a relevant argument (again rooted in Russell's writings).[21]

The argument goes like this: Compare the natural number series starting with 0 (as Frege did) and that starting with 1 (as Dedekind did). Clearly, they are different. But conceived of structurally, we seem to loose the difference. Dummett comments:

> The number 0 is not differentiated from the number 1 by its position in a progression, otherwise there would be no difference between starting with 0 and starting with 1. That is enough to show that we do not regard the natural numbers as identifiable solely by their positions within the structure comprising them (p. 52).

If this is correct, Dedekindian 'abstraction', or corresponding structuralist positions more generally, are simply incoherent. At the same time, Dummett acknowledges:

> Mathematicians frequently speak as if they did believe in such an operation. One may speak, for example of 'the' five-element non-modular lattice. There are, of course, many non-modular lattices with five elements, all isomorphic to one another; if you ask him which of these he means, he will reply, 'I was speaking of the abstract five-element non-modular lattice' (p. 52)

It appears, then, that Dedekind's and similar approaches accord with mathematical practice. How are we to deal with this recalcitrant fact?

Dummett's solution takes us back to another Russellian suggestion: '[E]ven if [the mathematician] retains a lingering belief in the operation of abstraction, his way of speaking is harmless: he is merely saying what holds good of any five element non-modular lattice' (ibid.). Later in his book, Dummett returns to this issue. He contrasts the position he sees as implicit in mathematical practice (that a mathematical theory 'always concerns all systems with a given structure') with Dedekind's position (that mathematics 'relates to abstract structures, distinguished by the fact that their elements have no non-structural properties'). He also notes that the former, labeled 'hard-headed structuralism' by him, was 'misattributed by Russell to Dedekind'. And with another rhetorical flourish, he dismisses the latter as 'mystical structuralism' (ibid.).

I have reserved Dummett's most original argument against Dedekind for last. It leads us back to his initial differentiation, and his corresponding evaluation, of our two thinkers:

> Dedekind approach to the question posed in his title [*Was sind und was sollen die Zahlen?*] differs utterly from Frege's. [...] Dedekind's treatment was that of a pure mathematician, whereas Frege was concerned with applications. Dedekind's central concern was to characterize the abstract structure of the system of natural numbers; what those numbers are used for was for him a secondary matter (p. 47).

Crucial in this passage is seeing Frege as 'concerned with applications'. For Dummett, this is 'a leading component of his general philosophy of mathematics' (p. 61). What is meant by 'application' in this context? For the natural numbers, it is their use as 'cardinal numbers'; for the reals, it is their use as 'measurement numbers'. Dummett is aware that these are not the only applications of the two number systems; but they are the 'salient' ones, those we should take as 'central to their definitions' (ibid.). In contrast, for Dedekind the question of application is 'external, an appendage which could have been omitted without damaging the theory as a whole' (p. 51). Dummett's core point is this: '[T]he general principles [of their application] belong to the essence of number, and hence should be made central to the way the numbers are defined or introduced' (p. 262).[22] That is why Frege's approach is seen as superior to Dedekind's.

Actually, for Dummett there is a second point at issue here as well, one that goes back to Frege's concern about how numbers are 'given to us', or about how we can 'recognize' and 'identify' them. The relevant Fregean question is as follows: Can this be done in purely structural terms? With respect to the real numbers, Dummett answers:

> Any system of objects having the mathematical structure of the continuum is capable of the same applications as the real numbers; but, for Frege, only those objects directly defined as being so applicable could be recognized as being the real numbers (p. 61).

And in connection with the natural numbers, he puts the same basic issue thus:

> [C]onstitutive of the number 3 is not its position in any progression whatever, or even in some particular progression, nor yet the result of adding 3 to another number, or of multiplying it by 3, but something

> more fundamental than any of these: the fact that, if certain objects are counted 'one, two, three', or equally, 'Nought, one, two', then there are 3 of them (p. 53).

Dummett is convinced that Frege is on the right track in this connection, and also that the issue really matters. In a final swipe at Dedekind, he adds (somewhat condescendingly): 'The point is so simple that it needs a sophisticated intellect to overlook it'. (ibid.)

4 Dedekind's Broader Reception and Defenses of Neo-Structuralism

At this point in our discussion, it may appear that Frege's superiority to Dedekind has been firmly established. His critics are sometimes too polemical, to be sure. Some of their arguments can also be disarmed fairly easily, at least if one reads Dedekind charitably. Still, a whole slew of other arguments remains. Surely, they are decisive, as is natural to assume. In this and the next section, I want to start turning the tables. This will involve considering several very different, much more positive responses to Dedekind's works. It will also lead to defenses of him against most, even if not all, of the criticisms mentioned so far.

A first point to observe here is that, while Frege has had numerous admirers within analytic philosophy, Dedekind's reputation has always been high among mathematicians and historians of mathematics – higher than Frege's, in fact. Dedekind made several lasting contributions to non-foundational parts of mathematic, and his foundational contributions have become firmly entrenched as well. The latter started with the impact his characterization of the natural numbers had on Giuseppe Peano and with the positive reception of his theory of chains by Ernst Schröder; it continued with David Hilbert's axiomatic approach to geometry, clearly inspired by Dedekind; and it reached a high point in Ernst Zermelo's and John von Neumann's generalization of his treatment of mathematical induction in transfinite set theory. In connection with set theory, another detail is noteworthy for present purposes: the way in which Dedekind's often maligned 'proof' of Proposition 66 influenced the form of the axiom of infinity in ZF set theory directly. And one can go on: to model theory (Dedekind's categoricity result, the idea of non-standard models), basic recursion theory (the focus on recursive functions), and other parts of logic.[23]

What about philosophy, however, especially in the analytic tradition? A development that is relevant, although somewhat indirectly, is the re-emergence of structuralist positions in the philosophy of mathematics during the last few decades. Two early instances were Paul Benacerraf's 'What Numbers Could Not Be' (1965) and Hilary Putnam's 'Mathematics without Foundations' (1967). But the defense of 'neo-structuralist' views really took

on steam in the 1980s, with publications by Michael Resnik, Stuart Shapiro, Geoffrey Hellman, Charles Parsons, and others. One outcome of their joint efforts was the differentiation between two versions of structuralism: 'eliminative structuralism', as represented by Hellman (following Putnam), and 'non-eliminative structuralism', as represented by Shapiro and others (partly following Benacerraf). People on both sides, while disagreeing in terms of their metaphysical convictions (nominalist or realist), also had a common motivation: dissatisfaction with the set-theoretic approach long dominant in the philosophy of mathematics.[24]

Crucially for present purposes, eliminative and non-eliminative structuralists alike claimed Dedekind as their distinguished forefather. Thus, Hellman writes:

> The idea that mathematics is concerned principally with the investigation of structures of various types in complete abstraction from the nature of individual objects making up those structures is not a novel one, and can be traced at least as far back as Dedekind's classic essay, 'Was sind und was sollen die Zahlen?' (Hellman, 1989, p. vii)

In Shapiro's main presentation of his structuralist position, we can read:

> A direct forerunner [of my position] is Dedekind. His development of the notion of continuity and the real numbers in [*Stetigkeit und irrationale Zahlen*], his presentation of the natural numbers via the notion of Dedekind infinity, in [*Was sind und was sollen die Zahlen?*], and some of his correspondence constitute a structuralist manifesto' (Shapiro, 1997, p. 14).

However, neither Hellman nor Shapiro is all that concerned about interpreting Dedekind accurately. The goal is, instead, to develop their respective versions of structuralism systematically. Besides a general a-historical attitude, part of the reason seems to be a deep-seated hesitancy to say more about Dedekind. Why? Because his specific philosophical views, as opposed to his general structuralist approach, are seen as flawed. It appears that the sustained attacks by Frege, Russell, and their followers have left their mark – especially the psychologism charge. This is also why, in debates with neo-Fregeans, it is sometimes Hilbert who is presented as the main alternative to Frege; neo-structuralist positions are then presented more as 'Hilbertian' than 'Dedekindian'.[25]

The degree to which some of Russell's criticisms influence later perceptions of Dedekind even among neo-structuralists can be illustrated further by two details. First, writers on both side of the structuralism divide, when trying to be charitable to Dedekind, follow Russell in associating a universalist, thus eliminativist, version of structuralism with him (the view that

arithmetic statements are about all progressions). This underlies Hellman's reference to Dedekind quoted above. Similarly, in Charles Parsons' recent defense of non-eliminative structuralism (Parsons 2007), Dedekind is used to introduce the eliminative alternative (even though Parsons is careful not to attribute that position to him in the end). Second, already in Benacerraf's influential article, "What Numbers Could Not Be," the author echoes Russell in doubting that there can be 'objects' with only structural properties; and once more, no further argument is provided. Even a committed 'realist' structuralist like Stuart Shapiro tends to call the relevant entities 'points' or 'positions' in a structure, rather than full-fledged 'objects', thus again siding with Russell.[26]

Having said that, the neo-structuralist literature contains arguments for why one should accept structurally conceived entities as 'objects' as well. Thus, Parsons (in the first chapter of his 2007 book) contrast a narrower conception of objecthood, rooted in the paradigm of medium-sized physical objects, with a broader 'logical conception' which he traces back to Quine's and Frege's works. [27] In terms of the latter, anything that is the referent of a singular term in truth-evaluable statements counts, basically, as an object. (A closely related 'logical' notion of object is defended by neo-Fregeans like Dummett, Wright, and Hale.) Parsons then connects the second, broader notion of object with his own considerations in favour of non-eliminative structuralism. At least implicitly, he thus provides a defense of Dedekind against this particular Russellian charge. He also brings Fregean and Dedekindian ideas together in a more positive way.

Parsons' book contains further defenses of a kind of structuralism similar to Dedekind's against some of the criticisms mentioned above.[28] Take Dummett's argument that, for a structuralist, the number series starting with 0 and that starting with 1 are indistinguishable, and if so, one is unable to differentiate between 0 and 1. Parsons responds as follows: It is true that the structuralist starts with a 'bare progression' (i.e., with *'a'*, *'succ'*, and *'N'* as the only basic symbols); and whether the base element *a* 'is' 0 or 1 is thereby not decided. But things get more determinate once we add one of two ingredients: we introduce addition recursively (thus determining whether *a* is neutral with respect to it or not); we use initial segments of the progression to measure the cardinality of finite sets (assigning the number *n* to a set either if it is equinumerous to $\{a, \ldots, n-1\}$ or to $\{a, \ldots, n\}$). Nevertheless, is Dummett not right that the initial setup does not determine whether the base element 'is' 0 or 1? Parsons' further response is this: In the kind of structuralism defended by him, the base element just is the base element; what gets added later is whether it plays the role of 0 or 1. As such, the position is entirely coherent.

This line of defense against Dummett's neo-Fregean criticism of Dedekind can be spelled out further, as follows: The two sides really have different aims. Frege and the neo-Fregeans, with their emphasis on application, try to analyze the concept of (finite) cardinal number (what Frege called 'Anzahl'). Structuralists, on the other hand, aim at characterizing one of the most basic mathematical structures, in terms of the closely related notions of 'simple infinity', 'progression', or 'model of the Dedekind-Peano axioms'. Considered as such, the two sides are compatible. A structuralist can even admit that the neo-logicist has, perhaps, finally succeeded in analyzing the notion of cardinal number. Nevertheless, for inner-mathematical purposes we can put that notion aside and use a different, more minimal one, namely that of a progression.

Additional defenses of Dedekind against the same Dummettian criticism are possible. Consider Frege and Dedekind on the natural numbers (the case of the reals is parallel). The thrust of Dummett's argument is that, while for Frege the core application of these numbers is 'built into' their very definitions, as it should be, for Dedekind it remains 'external'. But do the two approaches really differ so sharply in that respect? Dummett relies on the observation that Dedekind, after introducing his natural number structure, adds an explanation for how to apply it to measure the cardinality of finite sets – by using initial segments as tallies, via 1–1 mappability. However, doesn't Frege's approach contain a corresponding step? Namely, a class has cardinality *n* if it is contained in *n* as an element. Why is Dedekind's account of application 'external', while Frege's isn't? The only reason would seem to be that the element relation is privileged over the 1–1 mappability relation. But what is the justification for that?[29]

Perhaps the most basic defense of Dedekind against the same charge is the following: Dummett, like his neo-logicist followers, focuses on certain 'salient' applications of the natural and real numbers. They are seen as essential, as what needs to be built into their very definitions. However, both the natural and the real numbers have a variety of different applications. Why single out some of them? Once again, it is hard to see what a (principled and non-question-begging) reason would be. Indeed, if pushed just one step further, this line of thought can be turned into an argument in favour of Dedekind over Frege. Dummett himself remarks in the case of the reals (see above): 'Any system of objects having the mathematical structure of the continuum is capable of the same applications as the real numbers'. If so, it is arguably an advantage of Dedekind's approach that it focuses squarely on the 'structure'. All applications can then be studied based on it; no detour through some supposedly privileged one is necessary.[30]

5 More Defenses of Dedekind, while Broadening the Horizon Further

As we saw, neo-structuralists tend to be reluctant to defend Dedekind himself, while offering rebuttals to attacks on related versions of

structuralism. Are there any more direct defenses of Dedekind in the current literature? Yes, there are. But before turning to them, let me mention some 'pro-Dedekind' considerations that have been around longer, although we have to go beyond the analytic tradition to find them. The treatment of Dedekind by the neo-Kantian philosopher Ernst Cassirer provides a rich source for such considerations.

Already in one of Cassirer's earliest publications, the long article 'Kant und die moderne Mathematik' (1907), he displays a good appreciation, not only of Dedekind's technical results, but of his philosophical views as well. In Cassirer's book, *Substanzbegriff und Funktionsbegriff* (1910), he goes further, by arguing explicitly for the superiority of Dedekind's approach over Frege's and Russell's. And in Cassirer's later writings, related themes and further refinements appear. Three aspects of Cassirer's Dedekind reception are particularly noteworthy for us: his characterization of Dedekind's structuralist conception of mathematical objects; his defense of that conception against Russellian and Fregean criticisms; and his historically grounded argument that Dedekind's approach represents the culmination of a long development within mathematical science.[31]

With respect to Dedekind's basic approach to the natural numbers, Cassirer writes:

> [Dedekind showed that] in order to provide a foundation for the whole of arithmetic, it is sufficient to define the number series simply as the succession of elements related to each other by means of a certain order – thereby thinking of the individual numbers, not as 'pluralities of units', but as characterized merely by the 'position' they occupy within the whole series (Cassirer, 1907, p. 46, my translation).

Similarly concerning the reals, or more specifically, the irrational numbers:

> We thus see that, to get to the concept of irrational number, we do not need to consider the intuitive geometric relationships of magnitude, but can reach this goal entirely within the arithmetic realm. A number, considered purely as part of an ordered system, consists of nothing more than a 'position' (p. 49, my translation).

In these passages, Cassirer characterizes Dedekind's structuralist position very aptly (as a form of non-eliminative structuralism, to use current terminology). He also brings it into relief against earlier positions (the 'pluralities of units' view for the natural numbers, the appeal to 'intuitive geometric relationships of magnitude' for the real numbers). He thus places Dedekind's views into a certain historical context: the arithmetization of analysis and, more generally, the rise of 'pure mathematics' in the nineteenth century.[32]

Not only has Cassirer, in contrast to Russell, no problem in grasping Dedekind's structuralism, he also responds directly to one of Russell's corresponding objections:

> If the ordinal numbers are to be anything, then they must – so it seems – have an 'inner' nature and character; they must be distinguished from other entities by some absolute 'mark', the same way in which points are different from instants, or tones from colors. But this objection mistakes the real aim and tendency of Dedekind's formation of concepts. What is at issue is just this: that there is a system of ideal objects whose content is exhausted in their mutual relations. The 'essence' of the numbers consists in nothing more than their position. (Cassirer, 1910, p. 39)

What is it that underlies Russell's and other critics' resistance to a structuralist conception of mathematical objects? Cassirer also has an answer to that question, one that partly anticipates Parsons. The answer is that the critics have not let go of an old (Aristotelian) conception of object based on the notion of 'substance', while what is really needed in modern mathematics is a broader and 'function-based' conception. For Cassirer, the 'real aim and tendency' of Dedekind's work is precisely to provide the latter. Moreover, this is the main respect in which his approach is superior to Frege's and Russell's.

Concerning Dedekind's notion of 'abstraction' – and as a defense against the charge that it involves a form of psychologism – Cassirer observes this:

> [Dedekind's form of abstraction] means logical concentration on the relational system, while rejecting all psychological accompaniments that may force themselves into the subjective stream of consciousness, which form no constitutive moment of this system (Cassirer, 1910, p. 39).

The phrase 'logical concentration' in this passage indicates that Cassirer interprets Dedekindian abstraction as a logical procedure, not a psychological process. As such, it is quite different from the kind of 'abstraction' dominant from Aristotle through the British Empiricists to Mill and others in the nineteenth century, as Cassirer also notes. Moreover, he connects this point to the 'givenness' and the 'existence' of numbers: 'Givenness can here [...] mean nothing other than complete logical determinateness' (Cassirer, 1907, p. 49, my translation); and 'the "existence" of [a number] in Dedekind's sense is not intended to mean more than such determinateness: its "being" consists simply in its function of marking a [...] position' (ibid., fn. 26, my translation).

One especially valuable feature of Cassirer's reception of Dedekind is, once more, that he puts his structuralist position in the context of broader changes in nineteenth-century mathematics. With views like Dedekind's,

'mathematics is no longer – as it was thought of for centuries – the science of quantity and number, but henceforth encompasses all contents for which complete law-like determinateness and continuous deductive inner-connection is achievable' (Cassirer, 1907, p. 40, my translation). Even more broadly:

> Here we encounter for the first time a general procedure that is of decisive significance for the whole formation of mathematical concepts. Wherever a system of conditions is given that can be realized in different contents, we can hold on to the form of the system as an invariant, putting aside the difference of contents, and develop its laws deductively. In this way we produce a new 'objective' formation whose structure is independent of all arbitrariness [...]. (Cassirer, 1910, pp. 40)

According to Cassirer, both Frege and Russell made lasting contributions to the development at issue, by spelling out the logical frameworks in which the laws for various systems can be 'developed deductively'. But it was Dedekind who pushed our understanding of 'mathematical concepts' in a structuralist direction.

Cassirer also weights in on whether a Fregean and Russellian conception of numbers, as cardinal numbers, is superior to a Dedekindian conception, as ordinal numbers. His main argument in this connection is contained in the following remark:

> [I]t becomes evident that the system of the numbers as pure ordinal numbers can be derived immediately and without circuitous route through the concept of class [...]. The theory of the ordinal numbers thus represents the essential minimum that no logical deduction of the concept of number can avoid (Cassirer, 1910, p. 53, translation modified).

For Cassirer the question of superiority has, at bottom, to do with which approach – the Frege-Russell approach or Dedekind's – is more in line with the development and the desiderata of modern mathematics. And insofar as Dedekind aims at distilling out the 'essential minimum' required for doing arithmetic, his procedure is preferable.

One core part of Cassirer's reading of Dedekind is, as we just saw, to shift the focus of discussion to 'logical determinateness'. By doing that, Dedekind's position becomes defensible against the charge of psychologism. This also clarifies the sense in which it might be seen as a form of 'logicism' (pace Russell, Carnap, and others, but in line with Schroeder, Peirce, and Hilbert).[33] Having said that, Cassirer does not help much with respect to spelling Dedekind's logicism out more precisely, i.e., what we should take its basic laws to be. In other words, Frege's corresponding criticism is not addressed. Nor does Cassirer provide any grounds for rebutting the standard criticisms

of Dedekind's Proposition 66 (from Russell to Boolos and Dummett). While his take on Dedekind is subtle and rich in a number of respects, it is limited and incomplete in others.

Cassirer's reception of Dedekind is interesting in itself. It also anticipated, by several decades, two important developments in current history and philosophy of mathematics. First, Dedekind's role in the rise of modern mathematics has been elaborated further along historical lines. Second, Dedekindian 'abstraction' and the resulting structuralist conception have been defended more philosophically. Concerning the former, several recent investigations of the transformation of mathematics in the nineteenth century mention Dedekind very prominently.[34] This recognition is also reflected in new anthologies on the history of the philosophy of mathematics, e.g., William Ewald's collection, *From Kant to Hilbert: A Sourcebook in the Foundations of Mathematics*, Volumes I and II (1996). And it is in line with the turn to 'mathematical practice' in the philosophy of mathematics during the last decade or so. But these are all aspects too big to be pursued further here; I can only point to them in the present essay.[35]

For a recent, direct defense of Dedekind's notion of abstraction, developed in reaction to Dummett's work, we can turn to W.W. Tait's article, 'Frege versus Cantor and Dedekind: On the Concept of Number' (1997). Independently of Cassirer, Tait argues that 'Dedekind abstraction', as he calls it, has a logical core; and consequently, it is misinterpreted along psychologistic lines. The specifics of the structuralist conception of mathematical objects that results from it is further developed in my own article, 'Dedekind's Structuralism: An Interpretation and Partial Defense' (2003). The case for Dedekind's continuing relevance has thus been carried into current analytic philosophy.

My considerations in this and the previous section are not meant to show that Dedekind is immune to every criticism raised against him. The dialectical situation is surely more complex than that, thus requiring further attention. But what such considerations do establish, I would submit, is that some of the critics' polemics against Dedekind – as relying on 'one of the strangest pieces of argumentation in the history of logic' (Boolos), or as holding 'mystical' views (Dummett) – are uncharitable, superficial, and inadequate in themselves. They also indicate that the case for thinking of a Dedekindian approach as superior to a Fregean, at least in some respects, is not as hopeless as one might have thought. But let me close my discussion with a more conciliatory gesture. It brings us back to the criticism of Dedekind's approach perhaps voiced most often.

Dedekind's 'proof' of the existence of an infinite set, in Proposition 66 of his 1888 essay, has been challenged in a variety of ways (as relying on an inconsistent theory of classes, as bringing in non-logical, non-mathematical, or even psychologistic assumptions). Now, his position does, indeed, depend on securing such existence. If Dedekind's original approach doesn't

work, how else could we proceed? The second half of a remark by Russell in *Principles*, which we already encountered, provides a hint:

> There seems, in fact, to be nothing to choose, as regards logical priority, between ordinals and cardinals, except that the existence of the ordinals is inferred from the series of the cardinals. (Russell, 1903, p. 241)

As noted before, Russell still assumes at this point that his original construction of 'cardinals', in terms of equivalence classes, can be made to work. Russell's antinomy undermined that hope, as he then realized. His own remedy – the construction of cardinals in his ramified theory of types – leaves us with a different problem: the reliance on a non-logical assumption, i.e., (his version of) the axiom of infinity. Yet, with the advent of neo-logicism, an alternative has become available.

Recall that the core of the neo-logicist approach to the natural numbers is Frege's Theorem: the fact that we can prove the existence of an actual infinity based on Hume's Principle alone (in second-order logic). This result is sometimes presented as a decisive advantage of neo-logicism over other approaches, including Dedekind's.[36] But could we not see neo-logicism and Dedekindian structuralism as compatible, even complementary? What I have in mind is this: Why not adopt Frege's Theorem for establishing the existence of a simple infinity, and then add a structuralist notion of 'abstraction' to it? In other words, why not combine 'Frege abstraction' and 'Dedekind abstraction', also more generally?[37] One objection might be that, once the neo-Fregean approach is available, the structuralist side becomes superfluous. But we have encountered a response to that already: Dedekind abstraction distills out the 'essential minimum' needed for arithmetic; similarly for analysis. It thus provides an additional, distinctive benefit.[38]

6 Concluding Assessments

This essay started with the observation that Frege and Dedekind pursued closely related projects: to provide logicist foundations for the natural and real numbers. Frege acknowledged that fact but also formulated various criticisms of Dedekind. The chapter then traced the later reception of both Frege's and Dedekind's works. Frege's philosophical views were valued much more highly by several influential early analytic philosophers; they also experienced a remarkable revival since the 1950s. Even Frege's approach to the philosophy of mathematics has been resurrected since the 1980s. The reception of Dedekind was more negative. Outside of analytic philosophy, there were exceptions; and currently a general revival of interest in him is underway. One of my overall goals was to expose these trends and to provide clarification about what is at issue in them.

Several of Frege's main works contain sustained philosophical discussions, e.g., *Die Grundlagen der Arithmetik*. Frege also contributed to areas beyond the philosophy of mathematics, especially to the philosophy of language and mind. In contrast, Dedekind's publications were all in mathematics, including the foundations of mathematics, and the philosophical remarks in them are sparse. There is no mystery, then, why Frege received more attention within philosophy. Still, the relative neglect or continued dismissal of Dedekindian ideas in analytic philosophy is striking. I argued that it is rooted in the strong influence of Frege and Russell, whose criticisms of Dedekind keep getting recycled by their followers. Another of my goals was to reveal the relations between such criticisms, thus reassessing their appropriateness and relative weight.

While a barrage of attacks has been directed at Dedekind's philosophical views over the years, some defenses have been forthcoming as well. This started in Cassirer's writings from early in the twentieth century, which themselves are beginning to be rediscovered. But philosophers in the analytic tradition are now providing defenses of Dedekind as well, more or less directly. A third goal of mine was to collect these defenses, thus making them available more widely and juxtaposing them with the attacks. In addition, I put a suggestion for how to reconcile 'Frege abstraction' and 'Dedekind abstraction' on the table, thus calling into question the frequently assumed opposition between neo-logicism and neo-structuralism. While the proposals and arguments involved have not been analyzed in conclusive detail in this chapter, it should be somewhat more plausible now that not only Frege's but also Dedekind's approach can be partly rehabilitated. Many subtle questions and open problems remain, of course, in both cases.

A final point: Compared to the tendency of favouring Frege over Dedekind in the analytic tradition, Cassirer's assessment exemplifies a striking reversal. His opposite evaluation – his arguments that Dedekind's position is superior to Frege's and Russell's – is based on a combination of historical and philosophical considerations. In current philosophy of mathematics, the historical dimension is increasingly taken seriously as well, also with respect to its own history. This development is contrary to the a-historical, or even anti-historical, self-image that has long shaped analytic philosophy. The result should, over time, be a more balanced view. The present essay is offered as a step toward such balance. Insofar as it is successful, this means: Not only can Frege's and Dedekind's achievements be acknowledged more accurately now, also in relation to each other; the ways in which they played a role in the development of analytic philosophy becomes clearer as well, thus their philosophical legacies more transparent.[39]

Notes

1. Often, I will refer to Frege's and Dedekind's writings via their original titles and publication dates. In quotations, I will use references to their standard translations; cf. the bibliography.

2. There are also similarities in Frege's and Dedekind's backgrounds. In particular, both come out of the Göttingen mathematical tradition; cf. Ferreirós (1999), Chapters. I–II, and Tappenden (2006).
3. For a survey of Dedekind's contributions to mathematics, concerning both foundational and non-foundational areas (as well as their relationship), cf. Reck (2008).
4. For my interpretation of Dedekind's structuralist position about mathematical objects and his related methodology, cf. Reck (2003, 2008). For more on Frege's side, cf. Reck (2005).
5. For the development of Dedekind's views, cf. Sieg & Schlimm (2005). Dedekind does mention Frege's book, *Begriffsschrift*, in Dedekind (1890), as Ansten Klev has pointed out to me.
6. There is one possible exception to this lack of influence. Namely, Frege's adoption of a theory of classes as central to his project, in *Grundgesetze*, may reflect reading Dedekind (1888); cf. Sundholm (2001), p. 61.
7. For Dedekind's implicit use of AC, cf. Ferreirós (1999), p. 237; for Frege's 'logicist notion of class' (and its limits), cf. Ferreirós (2011); and for Dedekind on sets, cf. Ferreirós (forthcoming).
8. In a letter to the teacher Keferstein (Dedekind 1890), Dedekind makes clear that Proposition 66 is precisely supposed to play the role of insuring consistency (via satisfiability); cf. Reck (2003).
9. For more on this point, including some necessary refinements, cf. Reck (2003). Frege might respond: This is not an explicit part of Dedekind's theory; it is not clear how to formulate it generally as a law; and could it be regarded as a logical law then? Thus more is required for a full defense of Dedekind here.
10. For Dedekind's side—less often discussed than Frege's—cf. Sieg & Schlimm (2005), p. 121.
11. For Frege's response to the antinomy, cf. Reck (2005); for Dedekind's, see Reck (2003), fn. 9.
12. Cf. Ferreirós (1999), Chapter 7, section 6, more recently also Ferreirós (2009).
13. One has to be careful, however, as the term 'formalism' is ambiguous. For more on the Dedekind–Hilbert connection, cf. Reck (2003), Sieg & Schlimm (2005), and Ferreirós (2009).
14. Russell's 'principle of abstraction' involves going from an equivalence relation to the corresponding equivalence class. For 'Dedekind abstraction', cf. Tait (1997) and Reck (2003).
15. In Reck & Price (2000), the view Russell attributes to Dedekind is called 'universalist structuralism'. For more on why Russell's misses his mark in this connection, cf. again Reck (2003).
16. What seems to be going on here is that Russell, still early in his career, accepts something close to Meinong's distinction between 'being' (of entities) and 'existence' (of objects).
17. One partial exception to this trend is Van Heijenoort (1967), which contains one letter by Dedekind (and one to him by Cantor. However, even here the Frege part is much longer and more substantive.
18. Besides Wright (1983, 1997), Hale (1987, 2001), and Hale & Wright (2001), cf. Boolos (1998), Burgess (2005), Cook (2007), Demopoulos (1995), and Heck (2011).
19. *F* and *G* are supposed to be (sortal) concepts, and '#*F*s' means 'the number of *F*s'.
20. This aspect is highlighted in Demopoulos & Clarke (2007); but compare Antonelli (2010).

21. Russell uses a similar argument against Peano; cf. Russell (1903), p. 126.
22. In Wright's and Hale's later work, this is called 'Frege's Principle'; cf. Wright (1997).
23. For a survey of Dedekind's contributions, see again Reck (2008); for Peano, Schröder, Zermelo, and von Neumann, cf. Ferreirós (1999); for Hilbert, cf. Reck (2003), Sieg & Schlimm (2005), and Ferreirós (2009); and for model theory and related fields, cf. Awodey & Reck (2002).
24. For a survey of structuralist positions in current philosophy of mathematics, cf. Reck & Price (2000).
25. Cf. Shapiro (1996), who builds on a rational reconstruction of the Frege-Hilbert controversy.
26. Cf. the discussions of Benacerraf and Shapiro in Reck & Price (2000).
27. Discussions of such a logical notion of object are already central to Parsons (1983).
28. Parsons' version of structuralism is not identical with Dedekind's version but close to it. I intend to explore the similarities and differences further in a future publication.
29. This is a central part of the response to Dummett's criticisms of Dedekind in Tait (1997). Note also that the notion of 1–1 mappability is built centrally into Frege's approach as well.
30. For the argument that, from a mathematical point of view, no application of the natural and the real number system should be seen as privileged, cf. Stein (1987).
31. My discussion of Cassirer in this essay will have to be very brief and sketchy. I plan to address his relation to Dedekind more fully in future publications; cf. also Heis (2011).
32. For more on the 'pluralities of units' and 'magnitudes' views, including Frege's and Dedekind's reactions to them, cf. Reck (2005); concerning the rise of 'pure mathematics', cf. Ferreirós (2007).
33. In some recent surveys, such as Demopoulos & Clarke (2007), Dedekind is acknowledged again as a main logicist; but reservations often remain; cf Ferreirós (forthcoming).
34. Cf. Ferreirós (1999), Corry (2004), and Gray (2008), also parts of Ferreirós & Gray (2006).
35. For the philosophical turn to mathematical practice, cf. Mancosu (2008); for related studies of Dedekind, cf. Ferreirós (1999), Avigad (2006), Reck (2009), earlier also Stein (1987).
36. Cf. again Demopoulos & Clarke (2007), as well as the references in it.
37. This suggestion can also be found in Simons (1998).
38. Some neo-logicists have expressed sympathy for this kind of approach in the case of the real numbers; cf. 'Wright (1997); but see Hale (2001) for a less conciliatory, more austere attitude.
39. I am grateful to Jeremy Heis, Ansten Klev, and Clinton Tolley for comments on an earlier version of this essay. All the remaining mistakes and other problems should, as usual, be attributed to me.

References

Antonelli, Aldo (2010): 'The Nature and Meaning of Numbers', *The Journal of Philosophy* 62, 191–212.

Avigad, Jeremy (2006): 'Methodology and Metaphysics in the Development of Dedekind's Theory of Ideals', in Ferreirós & Gray (2006), pp. 159–186.
Awodey, S. & Reck, E. (2002): 'Completeness and Categoricity, Part I: 19th-Century Axiomatics to 20th-Century Metalogic', *History and Philosophy of Logic* 23, 1–30.
Benacerraf, Paul (1965): 'What Numbers Could Not Be', *Philosophical Review* 74; reprinted in Benacerraf & Putnam (1983), pp. 272–294.
Benacerraf, P. & Putnam, H., eds (1964/1983): *Philosophy of Mathematics: Selected Readings*, 1st ed., Prentice Hall, 1964; 2nd ed., Cambridge University Press, 1983.
Boolos, George (1998): *Logic, Logic, and Logic*, Harvard University Press.
Burgess, John (2005): *Fixing Frege*, Princeton University Press.
Carnap, Rudolf (1931): 'The Logical Foundations of Mathematics', *Erkenntnis*, 91–102; reprinted in Benacerraf & Putnam (1964), pp. 41–52.
—— (1949): *Meaning and Necessity*, University of Chicago Press.
Cassirer, Ernst (1907): 'Kant und die moderne Mathematik', *Kant-Studien* 12, 1–40.
—— (1910): *Substanzbegriff und Funktionsbegriff*, Bruno Cassirer: Berlin; English trans., by Swabey and Swabey, in *Substance and Function & Einstein's Theory of Relativity*, Open Court: Chicago, 1923.
Cook, Roy, ed., (2007): *The Arché Papers on the Mathematics of Abstraction*, Springer: New York.
Corry, Leo (2004): *Modern Algebra and the Rise of Mathematical Structures*, 2nd ed., Birkhäuser: Basel.
Dedekind, Richard (1872): *Stetigkeit und irrationale Zahlen*; reprinted in *Gesammelte Mathematische Werke, Vol. 3*, R. Fricke, ed., Vieweg, 1932, pp. 335–391; English trans., 'Continuity and irrational Numbers', in Dedekind (1963), pp. 1–27.
—— (1888): *Was sind und was sollen die Zahlen?*; reprinted in *Gesammelte Mathematische Werke, Vol. 3*, R. Fricke, ed., Vieweg, 1932, pp. 315–334; English trans., 'The Nature and Meaning of Numbers', in Dedekind (1963), pp. 29–115.
—— (1890): Letter to Keferstein, in Heijenoort (1967), pp. 98–103.
—— (1963): *Essays on The Theory of Numbers*, W.W. Beman, ed. and trans., Dover: New York.
Demopoulos, William, ed. (1995): *Frege's Philosophy of Mathematics*, Harvard University Press.
Demopoulos, W. & Clarke, P. (2007): 'The Logicism of Frege, Dedekind, and Russell', in *The Oxford Handbook of Philosophy of Mathematics and Logic*, S. Shapiro, ed., Oxford University Press, pp. 166–202.
Dummett, Michael (1973): *Frege: Philosophy of Language*, Harvard University Press.
—— (1991): *Frege: Philosophy of Mathematics*, Harvard University Press.
Ewald, William, ed. (1996): *From Kant to Hilbert: A Sourcebook in the Foundations of Mathematics*, Vols. I–II, Oxford University Press.
Ferreirós, José (1999): *Labyrinths of Thought: A History of Set Theory and its Role in Modern Mathematics*, Birkhäuser: Basel; 2nd ed., 2007.
—— (2007): 'The Rise of Pure Mathematics, as Arithmetic in Gauss', in *The Shaping of Arithmetic after C.F. Gauss's* Disquisitiones Arithmeticae, C. Goldstein, N. Schappacher & J. Schwermer, eds, Springer, pp. 235–268.
—— (2009): 'Hilbert, Logicism, and Mathematical Existence', *Synthese* 170, 33–70.
—— (2011): 'On Arbitrary Sets and ZF', *The Bulletin of Symbolic Logic* 17, 361–393.
—— (forthcoming): 'On Dedekind's Logicism', in *Analytic Philosophy and the Foundations of Mathematics*, A. Arana & C. Alvarez, eds, Palgrave: London.

Ferreirós, J. & Gray, J., eds (2006): *The Architecture of Modern Mathematics*, Oxford University Press.
Frege, Gottlob (1879): *Begriffsschrift*, Nebert: Halle; reprinted by Wissenschaftliche Buchgesellschaft: Darmstadt, 1977; English trans., *Begriffsschrift*, in Heijenoort (1967), pp. 1–82.
—— (1884): *Die Grundlagen der Arithmetik*, Koebner: Breslau; reprinted by Meiner, 1986, C. Thiel, ed.; English trans., by J.L. Austin, *The Foundations of Arithmetic*, 1950; reprinted by Northwestern University Press, 1980.
—— (1893/1903): *Grundgesetze der Arithmetik*, Vols. I and II, Pohle: Jena; reprinted by Olms: Hildesheim, 1998; English trans., in excerpts, *The Basic Laws of Arithmetic*, M. Furth, ed. and trans., University of California Press, 1964.
Geach, P. & Black, M., eds (1952): *Translations from the Philosophical Writings of Gottlob Frege*, Blackwell: Oxford.
Gray, Jeremy (2008): *Plato's Ghost: The Modernist Transformation of Mathematics*, Princeton University Press.
Hale, Bob (1987): *Abstract Objects*, Blackwell: Oxford.
—— (2001): 'Reals by Abstraction', *Philosophia Mathematica* 8, 100–123; reprinted in Hale & Wright (2001), pp. 399–420.
Hale, B. & Wright, C. (2001): *The Reason's Proper Study: Essays Towards a Neo-Fregean Philosophy of Mathematics*, Oxford University Press.
Heck, Richard (2011): *Frege's Theorem*, Oxford University Press.
Heijenoort, Jean van, ed. (1967): *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931*, Harvard University Press.
Heis, Jeremy (2011): 'Ernst Cassirer's Neo-Kantian Philosophy of Geometry', *British Journal for the History of Philosophy* 19, 759–794.
Hellman, Geoffrey (1989): *Mathematics without Numbers: Towards a Modal Structural Interpretation*, Clarendon Press: Oxford.
Mancosu, Paolo, ed. (2008): *The Philosophy of Mathematical Practice*, Oxford University Press.
Parsons, Charles (1983): *Mathematics in Philosophy*, Cornell University Press.
—— (2007): *Mathematical Thought and its Objects*, Cambridge University Press.
Reck, Erich (2003): 'Dedekind's Structuralism: An Interpretation and Partial Defense', *Synthese* 137:3, pp. 369–419.
—— (2005): 'Frege's Natural Numbers: Motivations and Modifications', in *Gottlob Frege: Critical Assessments of Leading Philosophers, Vol. III*, M. Beaney & E. Reck, eds, Routledge: London, pp. 270–301.
—— (2008): 'Dedekind's Contributions to the Foundations of Mathematics', *Stanford Encyclopedia of Philosophy*, E. Zalta, main editor; available online since April 22, 2008, pp. 1–32; updated and revised in 2011.
Reck, E. & Price, M. (2000): 'Structures and Structuralism in Contemporary Philosophy of Mathematics', *Synthese* 125, 341–383.
Russell, Bertrand (1903): *Principles of Mathematics*; reprinted by Routledge: London, 1992.
—— (1904): 'The Axiom of Infinity', *Hibbert Journal* 2, 809–812; reprinted in *Essays in Analysis*, D. Lackey, ed., Allen & Unwin: London, 1973, pp. 256–259.
—— (1919): *Introduction to Mathematical Philosophy*, Allen & Unwin: London.
Shapiro, Stuart (1997): *Philosophy of Mathematics: Structure and Ontology*, Oxford University Press.
—— (1996): 'Space, Number and Structure: A Tale of Two Debates', *Philosophia Mathematica* 4, 148–173.

Sieg, W. & Schlimm, D. (2005): 'Dedekind's Analysis of Number: System and Axioms', *Synthese* 147, 121–170.
Simons, Peter (1998): 'Structure and Abstraction', in *Philosophy of Mathematics Today*, Matthias Schirn, ed., Clarendon Press: Oxford, pp. 485–501.
Stein, Howard (1987): '*Logos*, Logic, and *Logistiké*: Some Philosophical Remarks on Nineteenth-Century Transformations of Mathematics', in *History and Philosophy of Modern Mathematics*, W. Aspray & P. Kitcher, eds, University of Minnesota Press, pp. 238–259.
Sundholm, Göran (2001): 'Frege, August Bebel and the Return of Alsace-Lorraine: The Dating of the Distinction between Sinn and Bedeutung', *History and Philosophy of Logic* 22, 57–73.
Tappenden, Jamie (2006): 'The Riemannian Background to Frege's Philosophy', in J. Ferreirós & J. Gray (eds), pp. 97–132.
Whitehead, A.N. & Russell, B. (1910–1913): *Principia Mathematica, Vols. I–III*, Cambridge University Press.
Wittgenstein, Ludwig (1921/22): *Tractatus Logico-Philosophicus*, Annalen der Naturphilosophie, 1921; English trans., Kegan Paul: London, 1922.
—— (1956): *Remarks on the Foundations of Mathematics*, Blackwell: Oxford.
Wright, Crispin (1983): *Frege's Conception of Numbers as Objects*, Aberdeen University Press.
—— (1997): 'On the Philosophical Significance of Frege's Theorem', in *Language, Thought and Logic*, R. Heck, ed., pp. 201–245; reprinted in Hale & Wright (2001), pp. 272–306.